Burridge-Knopoff model: Exploration of dynamic phases

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Slider-block models are often used to simulate earthquake dynamics. However, the models' origins are more conceptual than analytical. This study uses Navier's equations of an elastic bulk to derive a one-dimensional slider-block model, the Burridge-Knopoff model. This model exhibits a critical phase transition by varying the friction parameter. Accurate analytical estimates are made of event size limits for the small scale, large scale, and intermediate dynamic phases. The absence of large scale quasiperiodic delocalized events is noted for the parameter set investigated here. The time intervals between large scale events are approximately exponentially distributed for the system in its critical state, in agreement with the theory of nonequilibrium critical systems and earthquake dynamics.

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I. INTRODUCTION

The Burridge-Knopoff (BK) model is [1] a conceptual representation of an earthquake fault. The model consists of a linear array of N blocks, resting on a frictional surface, coupled to nearest neighbors by elastic springs with spring constant k_c, and each block connected via a leaf spring with spring constant k_t to a rigid plate moving at a constant velocity (see [2]). The area of the frictional surface in contact with the blocks is analogous to the fault. Elastic coupling between blocks represents the elastic nature of the lithosphere on the time scale of earthquake behavior. The constantly moving rigid plate is a representation of a tectonic plate, slowly moving relative to a neighbor plate, increasing the strain near the fault, the leaf springs.

Numerical studies by Carlson *et al.* [3–8] based on the BK model have observed a power law moment (event size) probability density distribution (PDD) in large agreement with earthquake behavior: the Gutenberg-Richter (GR) law [9]. A finitely driven homogenous BK model revealed that for the parameter regime investigated three distinct classes of events could be identified [5]: localized microscopic events, large but localized events, and delocalized events. The localized events obey a power law probability density distribution consistent with the GR law. In contrast, delocalized events cause a deviation from this power law with an excess of large scale events. These quasiperiodic [4] delocalized events, also observed in an infinitely slowly driven system [8], occur when the slip zone exceeds a delocalization length scale and the exponentially growing slip rates can no longer be retarded by the friction. It was noted that events begin to run into each other upon variation of the frictional parameter [5] and at the onset of this behavior a power law without large scale excess is observed. This power law, however, demonstrates a smaller exponent than is compatible with the GR law.

In addition, Vieira *et al.* [10] have reported a critical transition, albeit with large scale excess, from stick slip to continuous sliding through a critical point in a finitely driven system.

Transitions in stick slip systems are not only observed in the BK model. Distinct dynamic phases are noted in an *experimental* spring-block system by Johansen *et al.* [11], which can also produce power law and exponential moment PDDs. Exponential PDDs for the moment may also be seen upon analysis of the BK model [12], thought to be due to another transition tuned by varying the ratio k_c/k_t.

What follows is a detailed establishment of the BK model from the theory describing a continuous elastic medium in contact with a rigid frictional surface. Using the same friction law as Carlson *et al.* [8], estimates of the allowed range of event sizes for limits of the friction are calculated, corresponding to the dynamic phases observed.

II. NAVIER'S EQUATIONS AND THE BURRIDGE-KNOPOFF MODEL

The movement of the tectonic plates can be described by mechanical equations of motion. A summary of the derivation of equations of motion is now presented, showing how a variant of the BK model may arise from an elastic bulk medium in contact with a frictional surface.

The tectonic plate can be represented by a continuous elastic bulk in frictional contact with a rigid frictional surface, as envisioned by the BK model. The equations describing the elastic bulk are given by Navier's equations [13], Eq. (1)

$$\frac{\partial^2 \mathbf{u}(\mathbf{r},t)}{\partial t^2} = \left(\frac{\lambda+\mu}{\rho}\right)\nabla[\nabla\cdot\mathbf{u}(\mathbf{r},t)] + \left(\frac{\mu}{\rho}\right)\nabla^2\mathbf{u}(\mathbf{r},t) + \frac{\mathbf{f}}{\rho}, \tag{1}$$

where λ and μ are Lamé parameters, ρ is the mass density, $\mathbf{r}$ is the displacement vector, $\mathbf{u}$ is the material displacement

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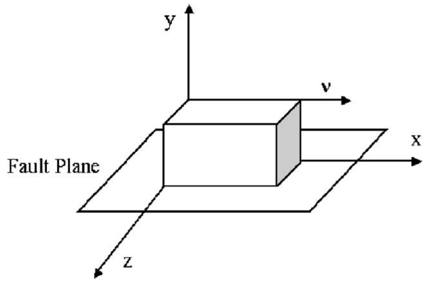

FIG. 1. Schematic of the earthquake system. ν is the velocity of the top surface of the bulk in the direction of the x axis.

from equilibrium, and $\mathbf{f}$ is a net body force, e.g., gravity, and t represents time. The assumptions made to obtain Eq. (1) are that the medium: is isotropic (no layering present in the bulk); is adequately modeled by a linear elastic constitutive law; has a constant density in space and time and the strain tensor is symmetric.

The Burridge-Knopoff (BK) model may arise as an approximation of these equations of motion. An earthquake system consists of a tectonic plate moving at a constant rate in relation to another plate. The interface between these two plates is described as the fault. If the two plates in contact are not moving in concert there will be a shear stress due to friction experienced on the fault plane. In order to model this system consider a configuration whereby a finite elastic bulk (tectonic plate) is in contact with a rigid frictional plane (fault) with the top surface of the elastic bulk moving with a velocity ν as in Fig. 1. The displacement $\mathbf{u}$ is a three-dimensional vector field and would be too computationally intensive to solve numerically for the purposes of this study, thus the equations need to be simplified. The approach to be taken involves first reducing the dimensionality of the problem. Here $\mathbf{r}$ has the orthogonal components x, y, and z, and the fault is defined as being the plane where $y=0$. The first assumption is that the displacement field is independent of z, which represents depth into the fault (see Fig. 1). With this assumption the model is representative of *large* earthquakes whose slip displacement at the fault spans the schizosphere. In addition, it is assumed that net body forces are negligible, $\mathbf{f}=\mathbf{0}$. The assumption is also made that slip only occurs in the direction of driving which will be defined as being the x direction, along the fault, thus $\mathbf{u}$ has the form

$$\mathbf{u} = (u(x,y,t),0,0).$$

Substituting this into Eq. (1) yields

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{\lambda + 2\mu}{\rho}\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\mu}{\rho}\frac{\partial^2 u(x,y,t)}{\partial y^2}. \tag{2}$$

It is desirable to obtain the solution on the fault itself, $y=0$. The bulk affected by the seismic motion is assumed to be thin such that strain is localized to a shearing zone within Δy of the fault plane, a property of any given fault. The boundaries are at the fault, $y=0$, and the edge of the shear zone $y=\Delta y$, the term $\frac{\partial^2 u(x,0,t)}{\partial y^2}$ may be simplified using a Taylor expansion of $u(x,\Delta y,t)$ leading to

$$\frac{\partial^2 u(x,0,t)}{\partial y^2} \approx \frac{2}{\Delta y^2}[u(x,\Delta y,t) - u(x,0,t)] - \frac{2}{\Delta y}\frac{\partial u(x,0,t)}{\partial y}. \tag{3}$$

The initial and boundary conditions of Eq. (4) are used,

$$u(x,\Delta y,t) = \nu t, \tag{4a}$$

$$\frac{\partial u(x,0,t)}{\partial y} = \frac{1}{2\mu}\Phi, \tag{4b}$$

$$\frac{\partial u(-L/2,y,t)}{\partial x} = 0, \tag{4c}$$

$$\frac{\partial u(L/2,y,t)}{\partial x} = 0, \tag{4d}$$

$$u(x,y,0) = g(x,y), \tag{4e}$$

where $\pm\frac{L}{2}$ are the boundaries in the x direction and, here, the function $g(x,y)$ is taken to be a spatially uncorrelated function with magnitude $|g(x,y)| \ll 1$. Φ is the shear stress at the boundary in contact with the frictional surface.

Taking each condition in turn, Eq. (4a) represents the bulk motion of the tectonic plate moving at a constant velocity ν. Equation (4b) implies simple shearing [13] where the shear strain is directly proportional to the shear stress friction at the boundary $y=0$ (the fault). Equations (4c) and (4d) define open boundary conditions at the boundaries $x=\pm\frac{L}{2}$, i.e., the elastic bulk does not extend past the boundaries. Finally, the initial condition of Eq. (4e) gives the initial displacement of all points.

Letting $u(x,0,t)$ be represented by $u(x,t)$, Eq. (2) at the boundary

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\lambda + 2\mu}{\rho}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + \frac{2\mu}{\rho \Delta y^2}[\nu t - u(x,t)] - \frac{\Phi}{\rho\Delta y}. \tag{5}$$

Equation (5) may be separated into two coupled differential equations

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = F - \frac{\Phi}{\rho\Delta y}, \tag{6a}$$

$$\frac{\partial F}{\partial t} = \frac{\lambda + 2\mu}{\rho}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) + \frac{2\mu}{\rho\Delta y^2}\left[\nu - \left(\frac{\partial u}{\partial t}\right)\right]. \tag{6b}$$

If the system defined by Eq. (6) is to be representative of earthquakes, stick slip behavior is expected, and so two time measures can be defined corresponding to each state: activity and quiescence. This can be achieved by defining the time t thus

$$t = \tau_a t_a = \tau_q t_q,$$

where τ_a and τ_q are the scales of the time measures t_a and t_q corresponding to activity and quiescence respectively.

Equation (6) may be nondimensionalized in each of the states (active, quiescent) using Eq. (7).

$$x = l_x x^*, \tag{7a}$$

$$y = y^* \Delta y, \tag{7b}$$

$$t = \tau_a t_a = \tau_q t_q, \tag{7c}$$

$$\nu = \frac{l_x}{\tau_q}, \tag{7d}$$

$$\sqrt{\frac{\lambda + 2\mu}{\rho}} = \frac{l_x}{\tau_a}, \tag{7e}$$

$$\sqrt{\frac{2\mu}{\rho}} = \frac{\Delta y}{\tau_a}, \tag{7f}$$

$$\frac{\partial u}{\partial t} = \frac{l_x}{\tau_a} e, \tag{7g}$$

$$F = \frac{l_x}{\tau_a^2} f, \tag{7h}$$

$$\Phi = \frac{\rho l_x \Delta y}{\tau_a^2} \varphi, \tag{7i}$$

where l_x is a length scale in the x dimension.

Rewriting Eq. (6) using Eq. (7) results in the nondimensional set of equations for quiescence, Eq. (8)

$$\frac{\tau_a}{\tau_q} \frac{\partial e}{\partial t_q} = f - \varphi, \tag{8a}$$

$$\frac{\partial f}{\partial t_q} = \frac{\tau_q}{\tau_a} \frac{\partial^2 e}{\partial (x^*)^2} + \left(1 - \frac{\tau_q}{\tau_a} e\right). \tag{8b}$$

This equation is written in terms of e and f. From Eq. (7g), e is the nondimensional velocity and from Eq. (7h), f is the nondimensional acceleration but can also be interpreted as the nondimensional shear stress which is given by

$$\frac{\rho \Delta y F}{\rho l_x \Delta y \tau_a^{-2}} = f.$$

During quiescence, $e=0$ everywhere thus $\frac{\partial^2 e}{\partial (x^*)^2} = 0$. In addition, the friction balances f such that there is no net shear stress, otherwise the velocity would become nonzero, and the quiescent time t_q becomes an inappropriate measure. Hence, for quiescence Eq. (8) reduces to

$$\frac{\partial f}{\partial t_q} = 1. \tag{9}$$

Repeating the process of nondimensionalization during activity in Eq. (6) using Eq. (7) (for activity $t = \tau_a t_a$) leads to

$$\frac{\partial e}{\partial t_a} = f - \varphi, \tag{10a}$$

$$\frac{\partial f}{\partial t_a} = \frac{\partial^2 e}{\partial (x^*)^2} + \left(\frac{\tau_a}{\tau_q} - e\right). \tag{10b}$$

For the slowly driven earthquake system, the time scale of activity τ_a is of the order of seconds and the time scale of quiescence τ_q is of the order of months/years. Hence, $\frac{t_q}{t_a} = \frac{\tau_a}{\tau_q} = \frac{\nu}{v_p} \sim 10^{-13} \ll 1$, taking the values for the plate drift $\nu \approx 10^{-9}\ \mathrm{ms}^{-1}$ and the compressive wave speed $v_p \approx 5 \times 10^3\ \mathrm{ms}^{-1}$. The limit $\frac{\tau_a}{\tau_q} \to 0$ leads to infinitely slow driving and the system, during activity, is described by Eq. (11)

$$\dot{e} = f - \varphi, \tag{11a}$$

$$\dot{f} = e'' - e. \tag{11b}$$

Here the dots represent differentiation with respect to nondimensional time during activity, t_a, and the dashes differentiation with respect to nondimensional space along the fault, x^*. Discretizing space leads to

$$\dot{e}_i = f_i - \varphi_i, \tag{12a}$$

$$\dot{f}_i = k_c(e_{i+1} + e_{i-1} - 2e_i) - e_i, \tag{12b}$$

where $k_c = \frac{1}{(\Delta x^*)^2}$, and the subscripts i refer to the ith grid point, normally referred to as a block.

Importantly, Eq. (12) is almost identical to the traditional BK model which can be written as follows:

$$\dot{e}_i = f_i - \varphi_i, \tag{13a}$$

$$\dot{f}_i = k_c(e_{i+1} + e_{i-1} - 2e_i) - k_t e_i. \tag{13b}$$

Note that Eq. (12) suggests that k_t of Eq. (13) should be such that $k_t = 1$.

In other studies the ratio k_c/k_t has been an important parameter for the system dynamics (see for example [12]). In a dimensional BK system this ratio is related to the shear modulus (or rigidity). In contrast, in the nondimensional system the shear modulus is a constant and is independent of the nondimensional numerical discretizations k_c and k_t. Note that this does not imply that any choices of discretizations are allowed. The system is being solved using a numerical method and so the choice of spatial and temporal discretizations must be made to achieve numerical stability. This is best accomplished by transmitting numerical information at equal speeds in both x^* and y^* dimensions.

In order for the numerical signal speeds, $\frac{\Delta x^*}{\Delta t_a}$ and $\frac{\Delta y^*}{\Delta t_a}$, to be the same the spatial discretization Δx^* is made equal to the shear zone width $\Delta y^* (=1)$ and so $k_c = 1$. The time step, Δt_a, is chosen such that some Courant-Friedrich-Lewy type condition is met, the details of which is dependent on the numerical method used to integrate the system of equations. When solving with the fourth order Runge-Kutta method, a useful guideline employed in simulations of the BK model here is to choose the time step such that $\Delta t_a \leq 0.1$.

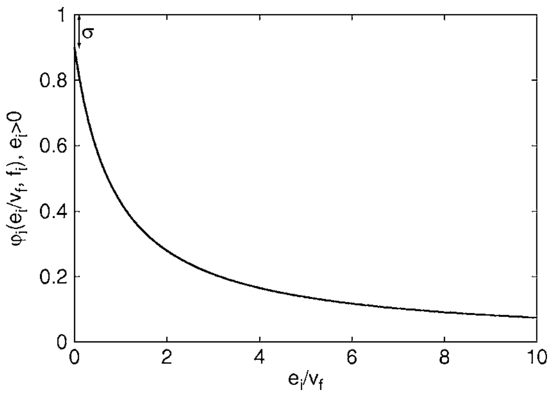

FIG. 2. The dynamic frictional function of Eq. (14) with $\sigma=0.1$.

The model as defined by Eq. (12) is readily solved by numerical techniques, but an understanding of the model's behavior in different parameter regimes is vital in understanding any emergent dynamics.

The frictional function used here (see Fig. 2) and by Carlson *et al.* [8] is given by Eq. (14), the Carlson and Langer (CL) friction law

$$\varphi_i\left(\frac{e_i}{v_f},f_i\right)=\begin{cases}\frac{(1-\sigma)}{[1+e_i/(1-\sigma)v_f]} & \text{if } e_i>0\\ (1-\sigma) & \text{if } e_i=0,\ f_i\geqslant 1\\ f_i & \text{if } e_i=0,\ f_i<1.\end{cases}\tag{14}$$

The frictional drop σ allows events to initiate abruptly with an acceleration proportional to σ, and eliminates a strong dependence of initial acceleration on the driving rate [8]. The parameter v_f is a characteristic velocity scale of the friction. Also, this friction law prohibits slip in the direction opposite to the driving.

Examining Eq. (12), if the system is relaxed, i.e., all $e_i=0$, then the system does not change and it is entering a quiescent period. The solution is started from an initially heterogeneous state in which the blocks are attributed random values of small magnitude for each f_i, corresponding to Eq. (4e). During quiescence the system is described by Eq. (9), which states that the shear stress over the entire system increases linearly in (quiescent) time t_q. This can only occur up to the time when an element of the system reaches the threshold of friction and the system becomes active. During a quiescent period the shear stress increases by an amount $\Delta=\varphi_i^{max}-\max(f_i)$ for all f_i [φ_i^{max} is the maximum of the frictional function and $\max(f_i)$ is the maximum shear stress in the system], i.e., the shear stress at all gridpoints is incremented by the amount necessary to make the gridpoint nearest to the slipping threshold move. The system then relaxes until all e_i are zero and this relaxation constitutes an event. The process of incremental shear stress increase and relaxation is continued for 10^6 iterations for a catalogue of 10^6 events. This is the typical event catalogue size for experimental results presented here.

III. ANALYTIC TREATMENT OF THE BK MODEL

A. Earthquake moment magnitudes in the BK model

The BK model is thought to be representative of the basic mechanics of earthquake events. An important measure used to characterize earthquakes is the moment M [9] representing the net slip along the fault during an earthquake event, defined by

$$M=\int_{t_0}^{t_0+\delta t}\int_A \mu|\mathbf{v}(\mathbf{r},t)|dA\,\mathrm{d}t,\tag{15}$$

where μ is the shear modulus [also in Eq. (1)], $\mathbf{v}$ is the velocity on the fault surface area A and the integrals are over the duration of the event δt and the surface of the fault. Using Eq. (7), the dimensional moment M in the model may be given by

$$M=\frac{\mu\Delta z}{2}\left(\Delta y\frac{v_p}{v_s}\right)^2 M_l,\tag{16}$$

where Δz is the schizosphere depth, v_p and v_s are the primary compressive wave speed and the secondary shear wave speed respectively, and the nondimensional moment M_l is given by

$$M_l=\int_D\sum_{i=1}^{N}e_i dt_a,\tag{17}$$

where the integral is over the event duration D and the sum is over the N gridpoints. Equation (16) allows the nondimensional moments measured from the model dynamics to be converted to the equivalent dimensional moment for a given fault, i.e., a fault with given depth, shear zone width, shear modulus, and wave speeds. The moment magnitude M_w may then be obtained through the relationship of Eq. (18) [14] where M is measured in newton meters

$$M_w=\tfrac{2}{3}\log_{10}|M|-6.\tag{18}$$

It is instructive to investigate the BK system's allowed maximum and minimum values of M_l. These limits may then be converted into estimates for the limits of M_w for a specific physical earthquake fault. When estimating M_w the following parameters for the fault geometry will be used:

$$\frac{v_p}{v_s}=\frac{1}{0.58},\tag{19a}$$

$$\mu=3\times10^{10}\ \text{Pa},\tag{19b}$$

$$\Delta z=15\ \text{km},\tag{19c}$$

$$\Delta y=1.5\Delta z.\tag{19d}$$

B. Dynamic phases and moment extrema

It may be readily seen from the equations of motion above, Eqs. (12) and (14), that the only nonlinearity is the frictional function φ_i. Under certain conditions this may be linearized and an analytical solution obtained. The approach that follows is similar to that of Carlson and Langer [5], but further to this, estimates are made of the limits of event moments for the regimes of v_f to be considered.

C. Small events in the limit $v_f\rightarrow\infty$

Equation (11) may be rewritten as

$$\left(\frac{\partial^2}{\partial t^2}+\frac{d\varphi}{de}\frac{\partial}{\partial t}+1-\frac{\partial^2}{\partial x^2}\right)e=0. \tag{20}$$

For the sake of brevity t_a and x^* are written as t and x, respectively. Consider the case of an event, centered at x_0, spanning a length l on the fault. If the average velocity on l is $\bar{e}$, then

$$\int_{x_0-l/2}^{x_0+l/2} e dx = l\bar{e}.$$

Integrating Eq. (20) over the length l leads to

$$\int_{x_0-l/2}^{x_0+l/2}\left(\frac{\partial^2}{\partial t^2}+\frac{d\varphi}{de}\frac{\partial}{\partial t}+1-\frac{\partial^2}{\partial x^2}\right)e dx=0,$$

or taking the partial differential operators outside the integral,

$$\left(\frac{d^2}{dt^2}+\frac{d\varphi}{de}\frac{d}{dt}+1\right)\int_{x_0-l/2}^{x_0+l/2} e dx-\left[\frac{\partial e}{\partial x}\right]_{x_0-l/2}^{x_0+l/2}=0.$$

Hence

$$\left(\frac{d^2}{dt^2}+\frac{d\varphi}{de}\frac{d}{dt}+1\right)\bar{e}-l^{-1}\left[\frac{\partial e}{\partial x}\right]_{x_0-l/2}^{x_0+l/2}=0. \tag{21}$$

If $e \ll v_f$, the dynamic friction, defined by Eq. (14), may be linearized as $\varphi=(1-\sigma)+e\frac{d\varphi}{de}=(1-\sigma)-e/v_f$ thus

$$\left(\frac{d^2}{dt^2}-v_f^{-1}\frac{d}{dt}+1\right)\bar{e}-l^{-1}\left[\frac{\partial e}{\partial x}\right]_{x_0-l/2}^{x_0+l/2}=0.$$

Assuming a triangular profile during an event with $e(x_0,t)=2\bar{e}(t)$ and $e(x_0\pm l/2,t)=0$ then the strain rate at $x=x_0\pm l/2$ may be calculated as the average of the strain rate in the positive and negative sense of the x axis about x, $\frac{\partial e^+}{\partial x}$ and $\frac{\partial e^-}{\partial x}$, respectively,

$$\frac{\partial}{\partial x}e(x_0\pm l/2,t)=\frac{1}{2}\left[\frac{\partial}{\partial x}e^{\pm}(x_0\pm l/2,t)+\frac{\partial}{\partial x}e^{\mp}(x_0\pm l/2,t)\right],$$

$$=\mp\frac{2\bar{e}}{l},$$

and so

$$\left(\frac{d^2}{dt^2}-v_f^{-1}\frac{d}{dt}+1+4l^{-2}\right)\bar{e}=0. \tag{22}$$

This can be rewritten in a standardized form

$$\frac{d^2\bar{e}}{dt^2}-2\alpha\frac{d\bar{e}}{dt}+\omega_l^2\bar{e}=0, \tag{23}$$

where $\alpha=\frac{1}{2v_f}$ and $\omega_l^2=(1+4l^{-2})$. Applying initial conditions of $\bar{e}=0$ and $\frac{d\bar{e}}{dt}=\sigma$ it follows that $\bar{e}$ is given by

$$\bar{e}=\frac{\sigma e^{\alpha t}}{\Gamma_l}\sin(\Gamma_l t). \tag{24}$$

Here $\Gamma_l=\sqrt{\omega_l^2-\alpha^2}$. The discriminant of Eq. (23) is Γ_l^2 and so in order for the system to support an event spanning infinite length ($l\to\infty$), and since $\Gamma_l^2\geq 0$ for oscillatory behavior, it follows that $v_f \geq 1/2$. It appears that there are constraints on the size of events supported by the system depending on v_f.

The event described by Eq. (24) will end at $t=\frac{\pi}{\Gamma_l}$ assuming the blocks all move in unison. This means that all kinetic energy is lost only to the boundaries of the slip zone to be stored as potential energy, and the friction, and therefore would be a maximum moment event. If Eq. (24) is integrated over the interval $t=\left[0,\frac{\pi}{\Gamma_l}\right]$ the average slip of a slipping block is obtained. The total moment of the event is $M_l=\int_0^{\pi/\Gamma_l} l\bar{e}dt$. Performing the integration

$$M_l^{max}=\frac{l\sigma}{\omega_l^2}\left[1+\exp\left(\frac{\pi\alpha}{\Gamma_l}\right)\right]. \tag{25}$$

If it is now considered that an event may consist of small sections of the fault, Δx^*, slipping in series, then the total moment is simply the sum of $n=l/\Delta x^*$ consecutive events

$$M_l^{min}=nM_\Delta=\frac{l\sigma}{\omega_\Delta^2}\left[1+\exp\left(\frac{\pi\alpha}{\Gamma_\Delta}\right)\right], \tag{26}$$

where ω_Δ and Γ_Δ indicates the values with $l=\Delta x^*$ for ω_l and Γ_l, respectively. In comparison with the event described by Eq. (25), Eq. (26) describes an event which loses energy in order to make the event propagate through the system in space and so it would be unlikely that large velocities would be attained unless the wave is amplified as it propagates. This indicates a minimum size event as here energy is also lost to initiating slip at the boundary of each section. In the limit of $v_f\to\infty$ (or $\alpha\to 0$) Eqs. (25) and (26) become

$$M_l^{min}=\frac{2l\sigma}{\omega_\Delta^2}, \tag{27a}$$

$$M_l^{max}=\frac{2l\sigma}{\omega_l^2}. \tag{27b}$$

D. Large events in the limit $v_f\to 0$

If the extreme of $v_f\to 0$ is now considered we know that the friction drops to 0 immediately as the block starts slipping. The dynamic friction is then linear and independent of e with $\varphi(e>0)=0$. Using this and Eq. (21), Eq. (28) follows:

$$\left(\frac{d^2}{dt^2}+1\right)\bar{e}-l^{-1}\left[\frac{\partial e}{\partial x}\right]_{x_0-l/2}^{x_0+l/2}=0. \tag{28}$$

Using the same reasoning as in the previous section the maximum event may be found by considering the motion over the event length l leading to Eq. (29)

$$\frac{d^2\bar{e}}{dt^2}+\omega_l^2\bar{e}=0. \tag{29}$$

Applying initial conditions of $\bar{e}=0$ and $\frac{d\bar{e}}{dt}=1$, the dynamic frictional drop with $v_f\to 0$, the solution for $\bar{e}$ may be found, being identical to Eq. (24) but with $\sigma=1$, $\alpha=0$, and $\Gamma_l=\omega_l$, and integrating over this event the maximum moment event

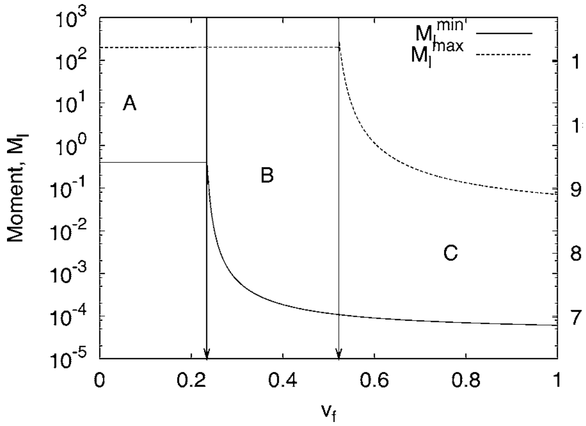

FIG. 3. Schematic of the limits on the moment M_l as a function of v_f with $l=100$, and $\sigma=10^{-4}$. Three regions (bounded by the arrows) may be observed as a large scale moment region A, a small scale moment region C, and an intermediate region B of moments spanning the small and large regions. The equivalent moment magnitudes M_w for a fault defined by Eq. (19) are also given.

spanning a length l may be obtained. Also by considering $n=l/\Delta x^*$ sections slipping consecutively in a single event the minimum moment is found.

As with Eq. (27) the event moment spanning a length l becomes

$$M_l^{min}=\frac{2l}{\omega_\Delta^2}, \tag{30a}$$

$$M_l^{max}=\frac{2l}{\omega_l^2}. \tag{30b}$$

Comparing Eqs. (30) and (27) it is apparent that there is significant difference between the scale of dynamics in the limits of $v_f\to 0$ and $v_f\to\infty$, the two being separated by a factor σ. These two limits represent distinct phases in the system's dynamics representing large scale and small-scale stick slip behavior for the two limits, respectively. The events in the limit $v_f\to\infty$ are controlled by the frictional drop σ [see Eq. (27)] which causes the events in this limit to tend to zero as $\sigma\to 0$. In the limit $v_f\to 0$ the events are not controlled by the frictional drop, only by the fault length slipping in an event. Figure 3 is a schematic of the moment limits suggested by Eqs. (25), (26), and (30) as a function of v_f for a fault with $l=100$, $k_t=1$, and $\sigma=10^{-4}$. Three regions are distinct in the figure with large scale (area A), small-scale (area C), and spanning (area B) moments. These correspond with the critical transition previously observed for the BK model using the formulation of Eq. (13) at $k_c=k_t=1$ in [2]. Note, due to the assumptions made, the limits of v_f for the different regions are merely estimates. Using Eqs. (16) and (18) along with the fault parameters given by Eq. (19), the equivalent moment magnitude may be determined as can also be seen in Fig. 3. Given the size of the fault the moment magnitudes are necessarily large: the smallest event in the model requires an event spanning the entire depth of the fault.

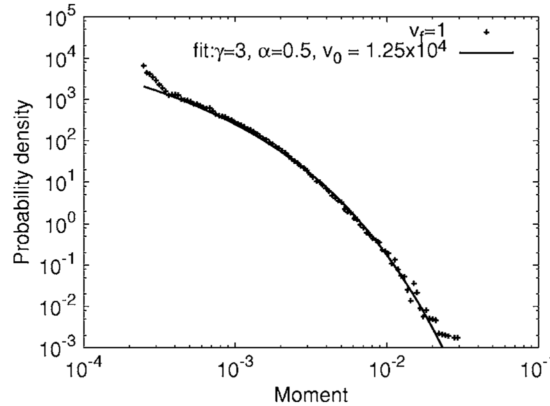

FIG. 4. The 100 block systems's moment PDD with $v_f=1$ is fitted by a stretched exponential function. The exponent, α, being 0.5 with the scale in the distribution, $v_0=1.25\times 10^{-4}$. The parameter γ readjusts the normalization of the probability density function.

It must be noted, however, that the above analysis does not give any information as to how often events of a certain length or moment may occur, i.e., the probability density distribution (PDD) of moments is still unknown. To determine such information it is necessary to resort to numerical techniques.

IV. DYNAMIC PHASE MOMENT DISTRIBUTIONS AND EARTHQUAKE RETURN TIMES

As mentioned the BK model was previously investigated [2] with $k_c=1$, and $k_t=1$ ($k=1$), and $\sigma=10^{-4}$. This study showed that the BK model undergoes a critical transition from large scale moments (stick slip) to small-scale moments (here called creep) with increasing v_f.

Here, the moment PDDs are further investigated for correspondence with the theory of Sec. III. Also, the moment PDD near criticality will be compared to results found by others, particularly Carlson *et al.* [4,8], in the regime $k_c>1$ where quasiperiodic delocalized events are observed. Note that single block events are neglected in the statistical analysis.

Figures 4–6 show the moment PDDs for three values of the frictional falloff v_f with $k_c=1$. The form and scale for the distributions changes. It is clear from these distributions that a transition is occurring with limits confirming the work presented in Sec. III.

For $v_f=1$ (Fig. 4) events have small moments spanning the theoretically predicted limits (see Sec. III C) of 1×10^{-4} for a two-block event to approximately 3×10^{-2} for a 55-block event; the largest number of blocks observed to slip in an event for the experimental data of Fig. 4. The moment PDD is fitted by a stretched exponential (SE) distribution with a normalization correction $\gamma=3$ [19].

With $v_f=0.1$ (see Fig. 5) the PDD is over a very different range of values and follows an *exponential* distribution. The theoretically predicted limits for moments in this large scale phase are approximately 0.8 and 200. Given the exponential nature of the moment PDD, events near the upper limit of

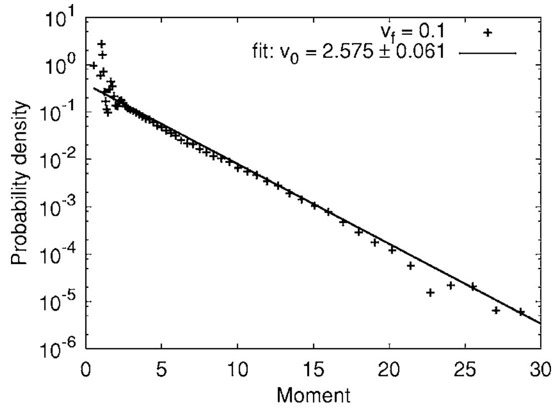

FIG. 5. Shown here is the moment PDD for the 100 block system on a log-linear scale. The moment PDD follows an exponential distribution with scale v_0=2.575±0.061.

200 are extremely rare and so have not been observed in simulations.

The transition between the two phases occurs at approximately $v_f = v_c$ and so large scale stick slip (LSSS) occurs for $v_f < v_c$ and creep for $v_f > v_c$. Section III D predicts a factor of approximately σ separating the two extremes of v_f and this is validated here.

It is worth noting that with small event size, as in creep behavior, one expects and observes a small decrease in the average shear stress in the system, i.e., the shear stress does not reduce much below the friction threshold in an event. This contrasts with LSSS behavior where large events relax the system far away from threshold. This difference between the two phases of LSSS and creep was the key to defining the order parameter of [2] to describe the transition.

For $v_f = v_c \approx 2/3$ a power law moment probability density distribution, with a small large scale excess is observed (see Fig. 6). To understand this transition near $v_f \approx v_c$ is more difficult than in the regime of LSSS or creep as the nonlinearity of the friction plays a key role in the dynamical behavior. It is, however, reasonable to suppose that there is a region, or point, where events may span across the full extent of both ranges of LSSS and creep behavior, as was suggested by theory (see Fig. 3) and seen in experimental data in Fig. 6. Equations (27) and (30) suggest that the smallest moment scale is controlled by σ and the largest by l, which is in turn restricted by the system size L. This indicates that in the limit of $\sigma \to 0$ when spanning behavior is observed the only constraint on the scale of moments is system size. This is to be expected of a system at or near a critical point.

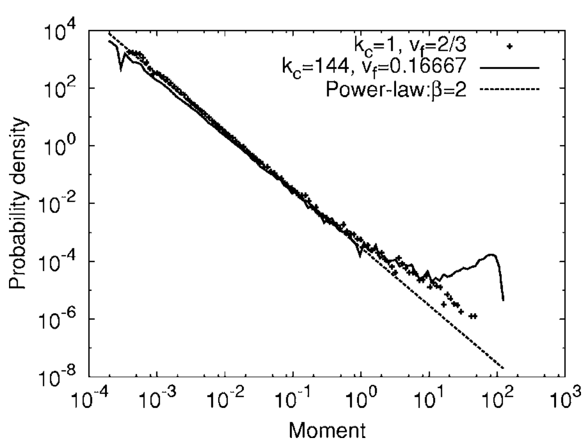

FIG. 6. This figure shows the moment PDD near the dynamic transition and where the large scale excess is present. For the dynamic transition that demonstrates no large scale excess $k_c=1$, $v_f=2/3$, and $\sigma=10^{-4}$. When $k_c=144$, $\sigma=10^{-2}$, and $v_f \approx 1/6$ the large scale excess is prominent.

Figure 6 shows the moment PDD for an $l=100$ system near the transition with $v_f=2/3$. In addition, the moment PDD of the system with $k_c=144$, and $v_f=1/6$ is plotted, which is the regime investigated by Carlson *et al.* [8] that exhibits delocalized large scale events. It is clear, however, that with $k_c=1$ there is no longer a sharp distinction between small and large scale regions. The power law nature of these large scale events suggest they are not delocalized events. Contrasting the two moment PDDs, delocalized events are not simply suppressed they are not present.

The large scale excess of delocalized events are quasiperiodic in nature [4]. This property of the delocalized events can be seen clearly from Fig. 7(a) where the PDD of shear stress intervals [20] Δf between delocalized events are shown. However, for the regime investigated here ($k_c=1$ and $\sigma=10^{-4}$) large events are not quasiperiodic. In fact the distribution of the shear force interval between events with moment greater than 0.5 is approximately exponential [Fig. 7(b)]. This suggests that these large scale events may obey a Poisson type process, which has been ascribed to large scale earthquake events [15]. The fitted scale to this distribution leads to the expectation value of Δf being given by $\langle \Delta f \rangle = 4.28 \times 10^{-2} \pm 9 \times 10^{-4}$. This is the expectation value of the shear stress interval between events with $M_l > 0.5$ ($M_w > 9.5$).In the terminology of seismology, this is called the return time. Equations (7) and (9) suggest that

$$\Delta t = \tau_q \Delta t_q = \tau_q \Delta f,$$

leading to

$$\langle \Delta t \rangle = \frac{\Delta y}{\sqrt{2}\nu} \frac{v_p}{v_s} \langle \Delta f \rangle .$$

This suggests that the return time for a fault with the properties given by Eq. (19) is approximately 37 000 yrs for earthquakes with moment magnitude of 9.5 or greater.

V. DISCUSSION

The derivation in Sec. II provides a direct link between the BK model and continuum mechanics, hitherto absent from the literature.

The BK model formulated here is a discretized wave equation. One must then be aware of numerical issues when obtaining any solution to the system. In what follows, such numerical concerns will be discussed. Following this, the results of infinitely slow driving obtained here will be compared to other results produced at a finite driving rate.

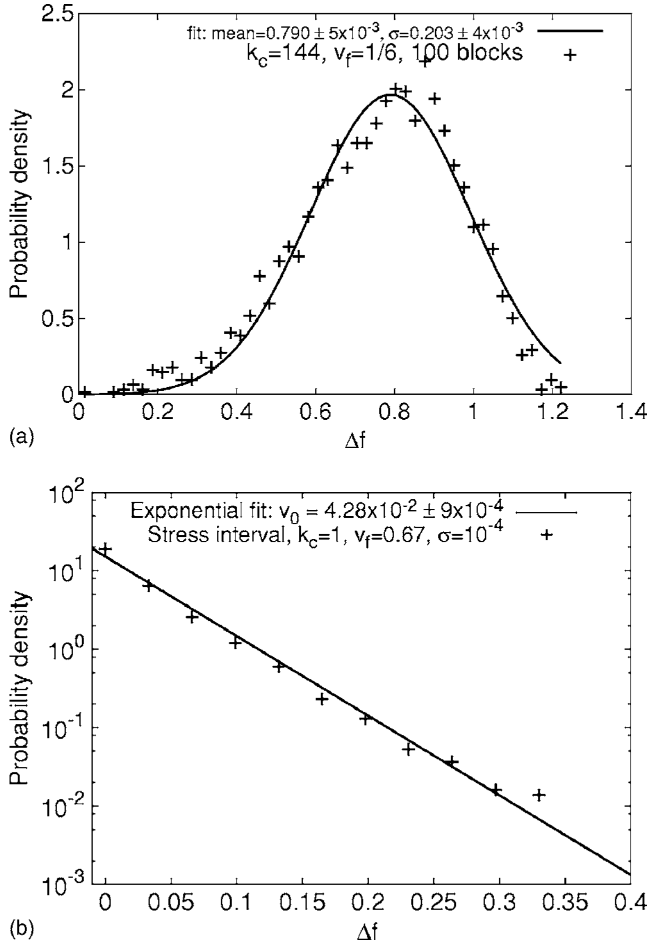

FIG. 7. (a) The delocalized events of the large scale excess are quasi-periodic in the sense of the shear force increments. Here the intervals between the 2509 events out of 10^6 with moment $\geqslant 10$ are recorded. (b) The intervals between events with moment $\geqslant 0.5$ deviating from the power law on the small scale of Fig. 6 are recorded. The fit to this distribution of intervals is exponential.

A. Delocalized events as a numerical instability

Previously, the BK model was shown to support a critical state. Of course, the most prominent feature of a system in a critical state is a lack of scale, emergent as power law behavior. Direct solution of the BK model in most published studies has largely found to result in a power law distribution of moments but with an additional large scale excess of events (Fig. 6) that occur quasiperiodically [Fig. 7(a)], termed delocalized events. While power law behavior supports the critical nature of the system, quasiperiodic events that are not power law distributed suggest a scale in the system's dynamics and as such the system is not scaleless and thus not critical. In addition, these delocalized events do not span the entire system for very large systems [8], indicating that the large scale excess does not scale with system size such that it shifts position, tending to infinity for an infinite size system. Results presented here suggest that these delocalized events are a numerical artifact because fixing the value of the spatial discretization k_c to a value where numerical stability may be expected, $k_c=1$, results in the delocalized events not being observed. The recurrence time PDD of events with large moments approximately follows an exponential distribution: the quasiperiodicity of large events is lost.

The case where $k_c=1$ has been investigated by Hähner and Drossinos [16] who studied a BK model implementing their own velocity dependent friction law but incorporating plastic deformation into the model. Plastic deformation allows aseismic slip to occur, relieving strain in the system, and an effective friction with memory results. power law behavior is observed and the system apparently exhibits tuned criticality without delocalized events as is observed here. In contrast, a smaller power law moment PDD exponent than expected of earthquake behavior was observed by them, 1.5, presumably arising from the different friction law. Other phases are not investigated in their work, however, and so cannot be commented on.

B. Numerical transitions

There have been other studies of transitions in the BK system but it is only those transitions associated with tuning the parameters of the friction that are considered relevant here. In [12], de Sousa Vieira noted a transition in the BK model by varying the ratio of numerical discretizations, k_c/k_t. However, in Sec. II it was argued on the grounds of numerical stability that the numerical signal speeds, $\frac{\Delta x^*}{\Delta t^*}$ and $\frac{\Delta y^*}{\Delta t^*}$, of the nondimensional equations of motion should be the same leading to the condition that $k_c/k_t=1$. This also reflects the "thin" geometry of the BK model where the shear strain of the elastic medium is localized near the fault as in real earthquake faults [17]. Further support for such a choice comes from results of the Olami-Feder-Christensen model [18], which is based on the BK model and shows the best correspondence with the observed power law moment PDD exponent of two when $k_c=k_t$. The "transition" observed varying k_c/k_t is considered to be a numerical artifact and, as such, not relevant. This is because it is expected that the solution to the system should be independent of the discretizations used, otherwise the solution may be unstable or prone to error. This is, in part, supported by the author herself who notes that the regime of the BK system demonstrating a partial power law with large scale excess has not been observed in experimental systems. However, de Sousa Vieira, by varying the numerical ratio to one, $k_c/k_t=1$, the value favored here, revealed behavior that *has* been observed in laboratory experiments of homogeneous systems: an exponential distribution of events sizes. From the above this is thought to be the true physical behavior of the system for the same friction parameters. If de Sousa Vieira had explored the parameter space further, a power law moment PDD without delocalized events would have been observed.

C. Stable and unstable motion?

As mentioned above, the BK model transition observed between the stick slip phase and creep phase has been investigated by Vieira *et al.* [10] utilizing the CL friction law but with a different value of $k_c/k_t>1$ and with a finite driving

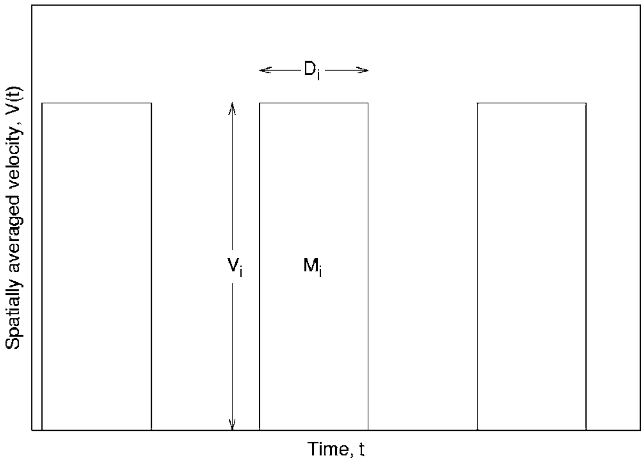

FIG. 8. Schematic of the total velocity in time for the unstable creep events.

velocity. Increasing the friction parameter v_f the system is seen to undergo a transition to continuous motion at $v_f \approx 1$. This coincides with the same transition point to creep behavior as found previously by us in [2] using Eq. (13). Vieira *et al.* describe this transition as, with initially high v_f, being in constant motion (fluid phase) and as the stick slip phase is approached by decreasing v_f, stationary states appear and begin to percolate through the system at the transition point. In what follows, it is shown that in addition to a fluid phase, a creep phase may also be observed at a finite but small driving rate.

Consider why the system may be in a fluid phase. Assuming there is no plastic deformation, a point on the bulk surface in contact with the fault must keep pace with the driving rate on average, otherwise there is a potential energy accumulation in the bulk. There are two classes of motion that allow the block to keep pace with the driving rate, ν, unstable (stick slip and creep) and stable (fluid). Fluid motion cannot be observed in the infinitely slowly driven system studied here. What follows seeks to establish a condition to determine whether stable or unstable motion is to be expected at a finite driving rate. Such a condition, could establish if creep and fluid motion may be observed as separate phases for the same driving rate.

The velocity of the system with depth Δz and shear modulus μ at a point in space x along the fault and in time t is given by $v(x,t)$. The average velocity of the system in space, over its entire length L, and time must be the same as the driving rate on average over a long time T, and so it follows that

$$\frac{1}{LT}\int_0^T\int_0^L v(x,t)dxdt = \frac{1}{T}\int_0^T V(t)dt = \langle V\rangle = \nu.$$

The stable fluid motion results in $V(t)=\nu$. The creep motion, however, must result in a time varying solution to $V(t)$ given its unstable nature. Slip events are simplified to have rectangular profiles in a velocity versus time plot (see Fig. 8), and so the ith event has a duration, D_i, and maximum velocity, V_i. The resulting slip from this event is proportional to the moment $M_i = \mu \Delta z L D_i V_i$. Thus it is expected that

$$\frac{1}{T}\sum_{i=1}^{N} D_i V_i = \frac{1}{\mu \Delta z L T}\sum_{i=1}^{N} M_i = \nu.$$

The average moment over the N events is $\frac{1}{N}\sum_{i=1}^{N} M_i = \langle M\rangle$. Thus

$$\frac{N}{\mu \Delta z L T}\langle M\rangle = \nu.$$

For a system operating at a point in parameter space in the creep phase, however, the moment distribution should be independent of the driving velocity ν assuming the driving is slow enough for events not to overlap, also indicating that the duration should be unchanged. It is then expected that the average moment $\langle M\rangle$ would also be independent of ν.

In a time T there can only be a finite number of creep events with average duration $\langle D\rangle$, otherwise it would be in fluid motion. This imposes the constraint for creep motion that the number of events N with duration $\langle D\rangle$ must be less than the time period T, or

$$N\langle D\rangle < T.$$

Hence

$$\nu L\langle D\rangle \mu \Delta z < \langle M\rangle. \tag{31}$$

For a system operating at a point in parameter space of the creep phase, varying the driving rate ν allows this condition to be met or not.

In the limit of infinitely slow driving the condition is automatically met if $\langle M\rangle$ is nonzero, which it is for the creep phase moment PDD. Note however, that from Eq. (27), $\langle M\rangle$ is proportional to σ in the creep phase. This dependence indicates that if $\sigma \to 0$ then $\langle M\rangle \to 0$ also: the condition for creep would not be met, suggesting a transition directly from fluid to large scale stick slip as found by Vieira *et al.* Yet experiments such as those carried out by Johansen *et al.* [11] indicate the existence of transitions from fluid to creep and then to stick slip in solid-solid friction dynamics. The above analysis suggests that with decreasing v_f, a finite nonzero value of σ and at an appropriately small driving rate there may be two transitions: the first from fluid to creep; the second from creep to stick slip as observed in experiment. A natural conclusion from this is that σ in the model here *should be nonzero*. However, with a faster driving rate, the system may undergo a transition directly from fluid to stick slip.

VI. CONCLUSIONS

The Burridge-Knopoff (BK) model has been derived from the theory describing an elastic bulk in contact with a rigid frictional surface showing that $k_t = 1$. Given the models origins from a wave equation, it has also been argued that to meet the Courant condition, k_c should also be equal to one. Previously, for the discretization $k_c = 1$, the BK model has been shown to exhibit a critical transition.

Analytical estimates were made of the limits of event moments in the large scale, small-scale, and critical phases. These estimates successfully reflect the differing scales between the large scale and small-scale phases. In addition, these estimates also correctly indicate the existence of a regime with event moments that span from the small-scale to the large scale phases' extreme limits. This does occur when the system is near the critical transition.

Contrary to results for the BK model with $k_c \gg k_t$, the BK model's power law moment probability density distribution (PDD) for $k_c = k_t = 1$ was found not to exhibit large scale quasiperiodic delocalized events. The PDD of intervals between large events was found to be an exponential distribution corresponding with behavior expected of critical systems and earthquakes.

ACKNOWLEDGMENT

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[19] A stretched exponential distribution of probability density $p(v)$ is given by $p(v)dv = \gamma\alpha\left(\frac{v}{v_0}\right)^{\alpha-1} e^{-(v/v_0)^{\alpha}} \frac{dv}{v_0}$ when $\alpha < 1$ [14]. The factor γ is a normalization constant, v_0 is a characteristic scale of v and α is the exponent associated with the stretched exponential.
[20] The interval between successive events is considered to be the increment in shear stress necessary to produce the next event as the driving is infinitely slow.